STABILITY OF DIFFERENCE APPROXIMATIONS TO DIFFERENTIAL EQUATIONS

by

Peter L. Falb *

Division of Applied Mathematics
Brown University
Center for Dynamical Systems
Providence, Rhode Island 02912

and

George M. Groome, Jr. **

Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

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13. ABSTRACT

Consider the differential equation (1) $\dot{x} = f(x)$ in a Banach space and let x^* be an equilibrium. The basic question treated is the following: if x^* is asymptotically stable and if (2) $x_{k+1} = x_k + h\varphi(x_k,h)$ is a one-step method, with fixed step size h, for integrating (1), then does the sequence x_k converge to x^*? It is shown that uniform asymptotic stability of (1) implies stability of (2) and that exponential asymptotic stability of (1) implies asymptotic stability of (2)

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Abstract

Consider the differential equation (1) $\dot{x} = f(x)$ in a Banach space and let x^* be an equilibrium. The basic question treated is the following: if x^* is asymptotically stable and if (2) $x_{k+1} = x_k + h\varphi(x_k,h)$ is a one-step method, with fixed step size h, for integrating (1), then does the sequence x_k converge to x^*? It is shown that uniform asymptotic stability of (1) implies stability of (2) and that exponential asymptotic stability of (1) implies asymptotic stability of (2).

1. Introduction.

Consider the differential equation

$$\dot{x} = f(x) \tag{1.1}$$

and let x^* be an equilibrium point. The basic question to be treated here is the following: if x^* is an asymptotically stable equilibrium and if

$$x_{k+1} = x_k + h\varphi(x_k,h) \tag{1.2}$$

is a one step method, with fixed step size h, for integrating (1.1), then does x_k converge to x^* as k tends to infinity? We shall show in our first main theorem that uniform asymptotic stability of (1.1) implies stability of (1.2) and in our second main theorem that exponential asymptotic stability of (1.1) implies asymptotic stability of (1.2) (improving a result of Skalkina [11]).

Our interest in the problem considered here stemmed from an investigation of iterative methods for solving the equation $F(x) = 0$ in a Banach space. If $f(x)$ is a function whose zeros include the zeros of F (for example, $f(x) = -(F_x')^{-1}F(x)$), then numerical integration of (1.1) will lead to iterates x_k corresponding to points $x(t_k;x_0)$ on the solution curve. If the initial point x_0 is in a region of attraction of the equilibrium x^*, then under what condition does x_k converge to x^*? Various

authors have used similar ideas to develop algorithms for solving $F(x) = 0$ ([1,2,3,5,10]) in particular situations. For example, Boggs ([1]) has integrated the equation $\dot{x} = -(F'_x)^{-1}F(x)$ with the A-stable methods of Dahlquist to generate iterates x_k which converge to a root of F. In [2,3,10], Euler and Runge-Kutta integration methods are used to generate iterates x_k which eventually lie within the region of convergence of Newton's method. Here, results are developed for general one step methods.

2. Uniform Asymptotic Stability.

Let X be a real Banach space with norm, $\|\cdot\|$, and let $S(r) = \{x \mid \|x\| \leq r\}$ be the closed ball of radius r about 0 in X. We let f be a mapping of X into itself and x^* be a zero of f. We assume, without loss of generality, that $x^* = 0$. Now, suppose that f is defined on the ball $S(R)$ and that $\varphi(x,h)$ is a mapping of $S(R) \times [0,h_0]$ into X. We assume throughout the sequel that the following conditions are satisfied:

(2.1) there are positive constants L and L' such that $\|f(x)-f(y)\| \leq L\|x-y\|$ and $\|\varphi(x,h)-\varphi(y,h)\| \leq L'\|x-y\|$ for all $x,y \in S(R)$ and $0 \leq h \leq h_0$;

(2.2) $\varphi(x,h)$ is uniformly continuous on $S(R) \times [0,h_0]$;

(2.3) $\varphi(x,0) = f(x)$ for all $x \in S(R)$; and

(2.4) $f(0) = \varphi(0,h) = 0$ for all $h \in [0,h_0]$.

We consider the differential equation

$$(2.5) \qquad \dot{x} = f(x)$$

and the one step integration method

$$(2.6) \qquad x_{k+1} = x_k + h\varphi(x_k,h).$$

[Note that (2.6) is consistent in view of the assumption (2.3)]. We now have

DEFINITION 2.7. The solution $x = 0$ of (2.5) is uniformly stable if, given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $\|x_0\| < \delta(\epsilon)$ implies that $\|x(t;t_0,x_0)\| < \epsilon$ for $t \geq t_0$ where $x(t;t_0,x_0)$ is the solution of (2.5) with $x(t_0;t_0,x_0) = x_0$. The solution $x = 0$ of (2.5) is uniformly asymptotically stable on a ball $S(r)$ if it is uniformly stable and if, given $\epsilon > 0$, there is a $T(\epsilon) > 0$ such that $\|x_0\| \leq r$ implies that $\|x(t;t_0,x_0)\| < \epsilon$ for $t \geq t_0 + T(\epsilon)$.

We note that since X may be infinite dimensional, uniform asymptotic stability and asymptotic stability are not equivalent ([9]).

We now assume that the solution $x = 0$ of (2.5) is uniformly asymptotically stable on the ball $S(R)$ for some $R > 0$. If $\delta(\epsilon)$ and $T(\epsilon)$ are the functions characterizing the stability of (2.5) as in definition 2.7, then we may assume that $\delta(\cdot)$ and $T(\cdot)$ are strictly monotonic continuous functions (see [7, p. 309]). We also suppose for simplicity that $t_0 = 0$ and we let $x(t;x_0) = x(t;0,x_0)$. We then have:

LEMMA 2.8. *Let* r, b *be real numbers such that* $0 < r \leq b \leq \delta(R)$. *Then there is a* $t_1 > 0$ *such that* $\inf\{\|x(t;x_0)\| \mid t \in [0,t_1],\ r \leq \|x_0\| \leq b\}$ *is strictly positive.*

Proof: Since $b \leq \delta(R)$, $\|x(t;x_0)\| \leq R$ and so $\|x(t;x_0)-x_0\| = \|\int_0^t \{f(x(s))-f(x_0)\}ds + tf(x_0)\| \leq L\int_0^t \|x(s)-x_0\|ds + t\|f(x_0)\|$ (where $x(\cdot) = x(\cdot;x_0)$). It follows from Gronwall's inequality and an integration by parts that

$$(2.9) \qquad \|x(t)-x_0\| \leq L\|x_0\| te^{Lt}$$

Therefore, $\|x(t;x_0)\| \geq \|x_0\| \ (1-Lte^{Lt})$ and we may choose $t_1 > 0$ such that $1 - Lt_1e^{Lt_1} > 0$.

Following Massera ([9]), we let $G(\cdot)$ be a continuous strictly increasing function with $G(r) \leq 2r$, $G(0) = 0$ and we introduce the Lyapunov function $V(\cdot)$ for (2.5) given by

$$(2.10) \qquad V(x_0) = \sup\{G(\|x(t;x_0)\|)(1+2t)/(1+t) \mid t \geq 0\}$$

for $0 \leq \|x_0\| \leq \rho$ where $\rho = \min\{1, \delta(R)\}$.

LEMMA 2.11. ([9]). $V(\cdot)$ *has the following properties:* (i) $G(\|x_0\|) \leq V(x_0) \leq 2\|x_0\|$; (ii) $\|V(x_0)-V(y_0)\| \leq M\|x_0-y_0\|$ *for some* $M > 0$; (iii) $\dot{V}(x_0) = \limsup_{k \to 0^+} [V(x(k;x_0))-V(x_0)]/k \leq -G(\|x_0\|)\{1+2T(\|x_0\|/2)\}^{-2}$; *and,* (iv) $V(x(k;x_0))-V(x_0) \leq -kG(\|x_0\|)\{1+2k+2T(\|x(k,x_0)\|/2)\}^{-2}$ *for*

$\|x_0\|, \|y_0\| \leq \rho$ and $k > 0$.

Letting $\psi(\|x_0\|) = G(\|x_0\|)\{1+2T(\|x_0\|/2)\}^{-2}$, we have:

LEMMA 2.12. *If* $0 < r < \delta(\rho)$ *and* $\epsilon > 0$, *then there is a* $k(r,\epsilon) > 0$ *such that*

$$V(x(k;x_0))-V(x_0) \leq k\{-\psi(\|x_0\|)+\epsilon G(\|x_0\|)\} \tag{2.13}$$

for $0 \leq k < k(r,\epsilon)$, $r \leq \|x_0\| \leq \delta(\rho)$.

Proof: Choose $t_1 > 0$ by lemma 2.8 so that $m = \inf\{\|x(t;x_0)\| \,|\, 0 \leq t \leq t_1, r \leq \|x_0\| \leq \delta(\rho)\} > 0$. Then $0 < m \leq \|x(t;x_0)\| \leq \rho$ for $0 \leq t \leq t_1$ and $r \leq \|x_0\| \leq \delta(\rho)$.

Since $A(k,\sigma) = [1+2k+2T(\sigma)]^{-2}$ is uniformly continuous on $[0,t_1] \times [m/2, \rho/2]$, there is an $\eta = \eta(\epsilon)$ such that $|A(k,\sigma')-A(0,\sigma)| < \epsilon$ if $|k| + |\sigma'-\sigma| < \eta$. Let $k(r,\epsilon)$ be the smaller of t_1 and the unique positive solution of $k + \frac{1}{2}L\delta(\rho)ke^{Lk} = \eta$. Letting $\sigma' = \|x(k;x_0)\|/2$ and $\sigma = \|x_0\|/2$, it follows from (2.9) that $|\sigma'-\sigma| \leq \|x(k;x_0)-x_0\|/2 \leq (L\delta(\rho)ke^{Lk})/2$ and hence, by virtue of lemma 2.11, that $V(x(k;x_0))-V(x_0) \leq -kG(\|x_0\|)A(k,\sigma') \leq k\{-\psi(\|x_0\|) + \epsilon G(\|x_0\|)\}$.

LEMMA 2.14. *There is an* $N > 0$ *such that, if* $x(t;x_0)$ *and* $x_0+hf(x_0)$ *are elements of* $S(R)$ *for* $0 \leq t \leq h \leq h_0$, *then* $\|x_0+hf(x_0)-x(h;x_0)\| \leq Nh^2\|x_0\|$.

Proof: Apply Gronwall's inequality.

LEMMA 2.15. *Let* $\rho = \min\{1, \delta(R)\}$ *and suppose that* $\varphi(x,h)$ *is uniformly*

<u>continuous on</u> $S(\rho) \times [0,h_0]$. <u>If</u> $0 < r < \delta(\rho)$, <u>then there is an</u> $h_1(r) > 0$ <u>such that</u>

$$V(x_0+h\varphi(x_0,h)) - V(x_0) \leq -\frac{1}{2}h\psi(r) < 0 \tag{2.16}$$

<u>whenever</u> $0 < h < h_1(r)$, $r \leq \|x_0\| \leq \delta(\rho)$.

<u>Proof:</u> Assume without loss of generality that $\delta(\rho) < \rho$. Then, if $h \leq (\rho-\delta(\rho))/[\delta(\rho)\max(L,L')]$, $x_0+h\varphi(x_0,h)$ and $x_0+hf(x_0)$ are elements of $S(\rho)$. Now, $V(x_0+h\varphi(x_0,h))-V(x_0) \leq |V(x_0+h\varphi(x_0,h)) - V(x_0+hf(x_0))| + |V(x_0+hf(x_0))-V(x(h;x_0))| + V(x(h;x_0)) - V(x_0)$. It follows from the previous lemmas, that, for $0 < r < \delta(\rho)$ and $\epsilon > 0$, there is a $k(r,\epsilon) > 0$ such that if

$$0 < h < h^* = \min(k(r,\epsilon),h_0,(\rho-\delta(\rho))/[\delta(\rho)\max(L,L')]) \tag{2.17}$$

then

$$\begin{aligned} V(x_0+h\varphi(x_0,h))-V(x_0) &\leq Mh\|\varphi(x_0,h)-f(x_0)\| + MNh^2\|x_0\| \\ &\quad - h\psi(\|x_0\|) + \epsilon G(\|x_0\|)h \end{aligned} \tag{2.18}$$

for $r \leq \|x_0\| \leq \delta(\rho)$.

Let $\alpha(h) = \sup\{\|\varphi(x,h)-\varphi(x,0)\| \mid r \leq \|x\| \leq \delta(\rho)\}$ and take $\epsilon \leq \psi(r)/(4G(\delta(\rho)))$. [Note that $\varphi(x,0) = f(x)$.] Since $\varphi(x,h)$ is uniformly

continuous, $\alpha(h)$ is continuous. Moreover, $\alpha(0) = 0$. Thus, the equation $\alpha(h) + Nh\delta(\rho) = \psi(r)/(4M)$ has a least positive root $\tilde{h} > 0$. If $0 < h_1(r) \leq \min(h^*, \tilde{h})$, then it follows that

$$(2.19) \qquad V(x_0 + h\varphi(x_0, h)) - V(x_0) \leq -\tfrac{1}{2}h\psi(r)$$

for $0 < h < h_1(r)$ and $r \leq \|x_0\| \leq \delta(\rho)$.

We can now prove the following:

THEOREM 2.20. <u>Suppose that the solution</u> $x = 0$ <u>of</u> (2.5) <u>is uniformly asymptotically stable on</u> $S(R)$. <u>Then, for any</u> $\epsilon > 0$, <u>there are</u> $h(\epsilon) > 0$ <u>and</u> $K(\epsilon) > 0$ <u>such that if</u> $\|x_0\| \leq G(\delta(\rho))/2$ <u>and</u> $0 < h < h(\epsilon)$, <u>then the solution</u> x_k <u>of</u> (2.6) <u>starting from</u> x_0 <u>satisfies the inequalities</u> (i) $\|x_k\| \leq \rho$ <u>for all</u> $k \geq 0$; <u>and</u> (ii) $\|x_k\| < \epsilon$ <u>for all</u> $k \geq K(\epsilon)/h$.

<u>Proof</u>: We may assume that $0 < \epsilon < \delta(\rho)$. Let $r = G(\epsilon)/4$ and let $h(\epsilon) = \min\{h_1(r), 1/L'\}$ where $h_1(r)$ is given by lemma 2.15. Also, let $K(\epsilon) = 2\{G(\delta(\rho)) - G(r/2)\}/\psi(r)$.

We consider three cases, namely: (i) $0 \leq \|x_0\| < r$, (ii) $r \leq \|x_0\| < 2r$, and, (iii) $2r \leq \|x_0\| \leq G(\delta(\rho))/2$.

<u>Case</u> (i): If $\|x_k\| < r$ for all $k \geq 0$, then $\|x_k\| < G(\epsilon)/4 \leq \epsilon/2 < \epsilon$ for all $k \geq 0$. On the other hand, if $\|x_k\| < r$ for $k = 0, 1, \ldots, n-1$ and $\|x_n\| \geq r$, then $\|x_n\| = \|x_{n-1} + h\varphi(x_{n-1}, h)\| \leq \|x_{n-1}\|(1 + hL') < 2r$ and we regard x_n as an initial point for case (ii).

<u>Case</u> (ii): We claim that $\|x_k\| < \epsilon$ for all $k \geq 0$. [Note that $\epsilon \leq \delta(\rho) \leq \rho$.] Clearly $\|x_0\| < 2r \leq \epsilon$. If $r \leq \|x_k\| < \epsilon$ for $0 \leq k \leq n$, then $G(\|x_{n+1}\|) \leq V(x_{n+1}) = V(x_0) + \sum_0^n (V(x_{k+1}) - V(x_k)) \leq V(x_0) - (n+1)h\psi(r)/2 < V(x_0) \leq 2\|x_0\| < 4r = G(\epsilon)$ by virtue of lemmas 2.11 and 2.15. Since G is strictly monotone, $\|x_{n+1}\| < \epsilon$ and the claim is established by induction.

Thus, combining cases (i) and (ii), we have shown that if $\|x_0\| < 2r$, then $\|x_k\| < \epsilon$ for all $k \geq 0$.

<u>Case</u> (iii): Clearly $\|x_0\| \leq G(\delta(\rho))/2 \leq \delta(\rho) \leq \rho$. Suppose that $r \leq \|x_k\| \leq \delta(\rho)$ for $k \leq n$. Then, $G(\|x_{n+1}\|) \leq V(x_{n+1}) = V(x_0) + \sum_0^n (V(x_{k+1}) - V(x_k)) \leq V(x_0) - (n+1)h\psi(r)/2 < 2\|x_0\| \leq G(\delta(\rho))$ by virtue of lemmas 2.11 and 2.15. Since G is strictly monotone, $\|x_{n+1}\| < \delta(\rho) \leq \rho$ and so, $\|x_k\| \leq \rho$ for all $k \geq 0$. Furthermore, if $(n+1)h \geq K(\epsilon)$, then $G(\|x_{n+1}\|) \leq V(x_0) - K(\epsilon)\psi(r)/2 \leq 2\|x_0\| - K(\epsilon)\psi(r)/2 \leq G(\delta(\rho)) - K(\epsilon)\psi(r)/2 = G(r/2)$. It follows that $\|x_{n+1}\| \leq r/2$. The theorem then follows from the first two cases.

We note that the theorem does not assert that the solution $x_k = 0$ of (2.6) is stable for fixed h. In other words, we do not claim that for given h and any $\epsilon > 0$ there is an $\eta = \eta(\epsilon, h)$ such that if $\|x_0\| < \eta$, then $\|x_k\| < \epsilon$ for all k. Bearing this in mind, we consider the following two-dimensional system:

$$
\begin{aligned}
\dot{x} &= y - x(x^2 + y^2) \\
\dot{y} &= -x - y(x^2 + y^2).
\end{aligned}
\tag{2.21}
$$

Let $V(x,y) = x^2+y^2$. Then $\dot{V}(x(t),y(t)) = -2(x^2(t)+y^2(t))^2$ along solutions of (2.21) and so, the trivial solution is uniformly asymptotically stable. If Euler's method is applied to (2.21), then the difference system

$$(2.22)\qquad \begin{aligned} x_{n+1} &= x_n + hy_n - hx_n(x_n^2+y_n^2) \\ y_{n+1} &= y_n - hx_n - hy_n(x_n^2+y_n^2) \end{aligned}$$

is obtained. Let $\Delta_h V(x,y)$ be given by

$$(2.23)\qquad \Delta_h V(x,y) = h^2(x^2+y^2)\left(x^2+y^2 - \frac{(1-(1-h^2)^{1/2})}{h}\right)\left(x^2+y^2 - \frac{(1+(1-h^2)^{1/2})}{h}\right)$$

so that $V(x_{n+1},y_{n+1}) - V(x_n,y_n) = \Delta_h V(x_n,y_n)$. Using (2.23), it is easy to verify that the trivial solution of (2.22) is not stable and that all solutions with $0 < h(x_0^2+y_0^2) < 1 + (1-h^2)^{1/2}$ $(0 < h < 1)$ are attracted to the invariant set $x^2+y^2 = (1-(1-h^2)^{1/2})/h$. Although the trivial solution of (2.22) is not stable for <u>fixed</u> h, the solutions of (2.22) can be made to remain arbitrarily close to zero by initially choosing h small enough. In other words, the theorem asserts that for given $\epsilon > 0$, there is an $h(\epsilon)$ such that if $h < h(\epsilon)$, then the solutions of (2.22) will lie within the ball $S(\epsilon)$.

3. <u>Exponential Asymptotic Stability</u>.

We now consider the case of exponential asymptotic stability.

DEFINITION 3.1. <u>The solution</u> $x = 0$ <u>of</u> (2.5) <u>is exponentially asymptotically</u>

stable on $S(r)$ if there are positive constants α, M such that $\|x(t;t_0,x_0)\| \leq M\|x_0\|e^{-\alpha(t-t_0)}$ for $\|x_0\| \leq r$ and $t \geq t_0$. Similarly, the solution $x_k = 0$ of (2.6) is exponentially asymptotically stable if there are positive constants b, h_1, M_1, β such that $\|x_k\| \leq M_1\|x_0\|e^{-\beta kh}$ for all $k \geq 0$ whenever $0 < h < h_1$ and $\|x_0\| \leq b/M_1$.

Skalkina ([11]) has shown that if the zero solution of (2.5) is exponentially asymptotically stable, then so is the zero solution of (2.6). We shall shortly present an improved version of his result.

LEMMA 3.2. If the solution $x = 0$ of (2.5) is exponentially asymptotically stable on $S(R)$, then the function $W(\cdot)$ defined by

$$W(x_0) = \sup(\|x(t;x_0)\|\exp(\arctan\ \alpha t) \mid t \geq 0) \tag{3.3}$$

for $x_0 \in S(R)$ has the following properties: (i) $\|x_0\| \leq W(x_0) \leq M\|x_0\|$; (ii) $|W(x_0)-W(y_0)| \leq K\|x_0-y_0\|$; (iii) $\dot{W}(x_0) \leq -c'W(x_0)$; and, (iv) $W(x(h;x_0)) - W(x_0) \leq -c\|x_0\|h$ for suitable positive constants K, c, c' (where α, N are the constants involved in the definition of exponential asymptotic stability).

Proof: Argue as in [7, pp. 309-311].

We now have

THEOREM 3.4. Suppose that the solution $x = 0$ of (2.5) is exponentially asymptotically stable on $S(R)$. Assume also that either (a) $\varphi(x,h) = f(x)$ or (b) $\varphi(x,h)$ is (Frechet) differentiable in x and $\varphi_x(x,h)$ is bounded and uniformly continuous on $S(R) \times [0,h_0]$. Then the solution

$x_k = 0$ <u>of</u> (2.6) <u>is exponentially asymptotically stable.</u>

<u>Proof:</u> Let $b \in (0,R)$ and let $\tilde{h} = \min(L^{-1}\log R/b, ((R/b)-1)/\max(L,L'), h_0)$. If $0 < h < \tilde{h}$, then $x + h\varphi(x,h)$ and $x + hf(x)$ are in $S(R)$ for $\|x\| \leq b$ and $\|x(t,x_0)\| \leq R$ for all $t \geq 0$ if $\|x_0\| \leq b$ (as $\|x(t,x_0)\| \leq \|x_0\| e^{Lt}$).

Now, let $a(h) = 0$ or $\sup\{\|\varphi_x(x,h) - \varphi_x(x,0)\| \mid \|x\| \leq b\}$ according as hypothesis (a) or (b) holds. If hypothesis (a) holds, then $|W(x+h\varphi(x,h)) - W(x+hf(x))| = 0 \leq Kha(h)\|x\|$. On the other hand, if hypothesis (b) holds, then $|W(x+h\varphi(x,h)) - W(x+hf(x))| \leq Kh\|\varphi(x,h) - \varphi(x,0)\| \leq Kh\|\varphi(x,h) - \varphi(0,h) - \varphi(x,0) + \varphi(0,0)\| \leq Kh \int_0^1 \{\|\varphi_x(tx,h) - \varphi_x(tx,0)\| \|x\|\} dt \leq Kha(h)\|x\|$. [Note that $\varphi(0,h) = \varphi(0,0) = 0$.] In other words, we always have

$$(3.5) \qquad |W(x+h\varphi(x,h)) - W(x+hf(x))| \leq Kha(h)\|x\|$$

for $\|x\| \leq b$.

Let h' be the least positive root of $K[a(h) + Nh] = c/2$ and let h_1 be any positive number with $h_1 < \min(\tilde{h}, h', 2M/c)$. If $h < h_1$ and $\|x_0\| \leq b$, then $W(x_0+h\varphi(x_0,h)) - W(x_0) \leq |W(x_0+h\varphi(x_0,h)) - W(x_0+hf(x_0))| + |W(x_0+hf(x_0)) - W(x(h;x_0))| + W(x(h;x_0)) - W(x_0) \leq h\|x_0\|\{Ka(h)+KNh-c\} \leq -h\|x_0\|c/2$.

Now let $M_1 = M$ and $\beta = c/(2M)$. We shall show by induction that if $\|x_0\| \leq b/M$, then

$$(3.6) \qquad \|x_k\| \leq M\|x_0\| e^{-\beta kh}$$

for all k. Clearly (3.6) holds for $k = 0$ and so, we suppose it holds for $0 \leq k \leq n$. For any such k, $\|x_k\| \leq M\|x_0\|e^{-\beta kh} \leq M\|x_0\| \leq b$ and so, $W(x_{k+1}) - W(x_k) \leq -h\|x_k\|c/2$. Since $W(x_k) \leq M\|x_k\|$, $W(x_{k+1}) \leq W(x_k)(1-\beta h)$. If $W(x_k) = 0$ for any $k \leq n$, then $W(x_{k+\ell}) = 0$ for all $\ell \geq 0$ and (3.6) is satisfied. Otherwise, $\|x_{n+1}\| \leq W(x_{n+1}) \leq W(x_n)(1-\beta h) \leq W(x_0)(1-\beta h)^{n+1} \leq M\|x_0\|e^{-\beta h(n+1)}$. Thus, the theorem is established.

References

[1] Boggs, P. T., The solution of nonlinear systems of equations by A-stable integration techniques, to appear.

[2] Bosarge, W. E., "Infinite Dimensional Iterative Methods and Applications", Ph.D. Dissertation, Brown University, Providence, 1969.

[3] Brown, K. R. and Johnson, G. W., Rapid computation of optimal trajectories, IBM J. of Res. and Dev., 11 (1967), 373-382.

[4] Falb, P. L. and deJong, J. L., "Some Successive Approximation Methods in Control and Oscillation Theory", Academic Press, New York, 1970.

[5] Gavurin, M. K., Nonlinear functional equations and continuous analogs of iterative methods, (Russ.) Izv. Vyss. Ucebn. Zaved. Mathematica, 6 (1958), 18-31; English trans., M. Kocho and J. H. Avila, eds., University of Maryland Tech. Rpt. 68-70, 1968.

[6] Hahn, W., "Theory and Application of Liapunov's Direct Method", Prentice Hall, Englewood Cliffs, 1963.

[7] Hale, J. K., "Ordinary Differential Equations", Wiley-Interscience, New York, 1969.

[8] Henrici, P., "Discrete Variable Methods for Ordinary Differential Equations", Wiley, New York, 1962.

[9] Massera, J. L., Contributions to stability theory, Ann. of Math., 64 (1956), 182-206.

[10] Meyer, G. H., On solving nonlinear equations with a one-parameter operator imbedding, SIAM J. Numer. Anal., 5 (1968), 739-752.

[11] Skalkina, M. A., On the preservation of asymptotic stability in transition from differential equations to the corresponding difference equations, (Russ.), Dokl. Akad. Nauk SSSR, 104(1955), 505-508.